Unimodular Equivalence of Graphs*

Russell Merris

Department of Mathematics and Computer Science
California State University
Hayward, California 94542

Submitted by Richard A. Brualdi

ABSTRACT

Let G be a simple graph with n vertices and m edges. Denote by Q the vertex-edge incidence matrix corresponding to some orientation of the edges of G. Define $L(G) = QQ^t$ and $K(G) = Q^tQ$. Then $L(G)$ is the so-called Laplacian matrix and $K(G)$ its edge version. Viewed as integer matrices, the Smith normal form of $L(G)$ is complicated, but the Smith normal form of $K(G)$, for connected graphs, is always $I_{n-2} \dotplus (n) \dotplus 0_{m-n+1}$. An edge version of the matrix-tree theorem is used in the proof. There is an application to digraph flows with constant resultants over abelian groups. Finally, if $L(G)$ and $L(H)$ are congruent, then G and H have the same chromatic polynomial.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. For each edge $e_k = \{v_i, v_j\}$ choose one of v_i or v_j to be the positive end of e_k and the other to be the negative end. Thus we produce a *orientation* of G. The vertex-edge incidence matrix $Q = Q(G)$ afforded by a fixed but arbitrary orientation of G is the n-by-m matrix (q_{ij}) given by

$$q_{ij} = \begin{cases} +1 & \text{if } v_i \text{ is the positive end of } e_j, \\ -1 & \text{if } v_i \text{ is the negative end of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

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While Q depends on the orientation of G, the *Laplacian matrix* $L(G) = QQ^t$ does not. For any orientation of G, $L(G) = D(G) - A(G)$, the difference of the diagonal matrix of vertex degrees and the adjacency matrix. Also known as the "Kirchhoff matrix" [11], or "matrix of admittance" [4], $L(G)$ first occurred in the celebrated matrix-tree theorem: If L_{ij} is the submatrix of $L(G)$ obtained by deleting row i and column j, then $(-1)^{i+j}\det L_{ij}$ is the number of spanning trees in G.

The *edge version*, $K(G) = Q^tQ$, depends on the orientation for the signs of its off-diagonal entries. Apart from these signs, $K(G) = 2I_m + A(\tilde{G})$, where $\tilde{G}$ is the line graph of G.

It is easy to see that $L(G)$ is a singular M-matrix whose nonzero eigenvalues are shared by the positive semidefinite matrix $K(G)$. These kinds of matrix-theoretic properties have been usefully exploited in the recent literature. (See, e.g., [5].) In this note, we shall be taking a somewhat different perspective, viewing $K(G)$ and $L(G)$ as integer matrices. In this context, for example, the matrix-tree theorem shows that every $(n-1)$-by-$(n-1)$ submatrix of the Laplacian matrix of a tree is unimodular (i.e., an integer matrix of determinant ± 1). In [1], K. A. Berman used the Smith normal form of $L(G)$ to completely characterize "bicycles of G over abelian groups." (See Section 5.) As a consequence of our Theorem 1, we are able to give an analogous characterization of digraph flows with constant resultants over abelian groups.

2. UNIMODULAR EQUIVALENCE

Two integer matrices A and B are *equivalent* if there exist unimodular matrices E and F such that $B = EAF$. Our vocabulary follows [9] with the following modifications. If G is a graph, denote by $d_k(G)$ the greatest common divisor of the determinants of all the k-by-k submatrices of $L(G)$. It is convenient to define $d_0(G) = 1$. The (integer) *invariant factors* of $L(G)$ are $s_k(G) = d_k(G)/d_{k-1}(G)$, $1 \leqslant k \leqslant n$. The Smith normal form of $L(G)$ is the n-square diagonal matrix $S(G)$ whose (k, k) entry is $s_k(G)$. From the matrix-tree theorem, $d_{n-1}(G)$ is the number of spanning trees in G. Thus, a necessary condition for two graphs to be Laplacian equivalent is that they have the same number of spanning trees.

EXAMPLE 1. Some additional elementary observations are:

(1.1) Since the rank of $L(G)$ is $n - C$, where C is the number of connected components of G, only the first $n - C$ invariant factors are nonzero:

(1.2) $S(G) = I_{n-1} \dotplus (0)$ if and only if G is a tree:

(1.3) if G has a square-free number t of spanning trees, then $S(G) = I_{n-2} \dotplus (t) \dotplus (0)$:

(1.4) on the other hand, for any $n > 2$, $S(C_n) = I_{n-2} \dotplus (n) \dotplus (0)$, where C_n is the simple circuit on n vertices;

(1.5) at the other extreme, the Smith normal form of the complete graph is $S(K_n) = (1) \dotplus nI_{n-2} \dotplus (0)$.

For the complete bipartite graph, $L(K_{s,t})$ is equivalent to $I_2 \dotplus sI_{t-2} \dotplus tI_{s-2} \dotplus (st) \dotplus (0)$, s. $t \geqslant 2$. However, this cannot be its Smith normal form (in general), because the divisibility conditions are not satisfied. Recall that the *elementary divisors* of $L(G)$ are the prime power factors of its invariant factors. The multiset of these elementary divisors is denoted by el(G): we will refer to them as the elementary divisors of G. It follows from the general theory [9, p. 30] that the elementary divisors of a diagonal matrix are the prime power factors of its diagonal entries. Thus, for example, $S(K_{3,4}) = \mathrm{diag}(1, 1, 1, 3, 12, 12, 0)$. [Note that el($G$) is empty if and only if G is a forest.]

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of n_1 and n_2 vertices, let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. A *coalescence* of G_1 and G_2 is any graph on $n_1 + n_2 - 1$ vertices obtained from $G_1 \cup G_2$ by identifying (i.e., "coalescing" into a single vertex) a vertex of G_1 and a vertex of G_2. For convenience, we denote by $G_1 * G_2$ any of the $n_1 n_2$ coalescences of G_1 and G_2.

We begin with some elementary observations, the first of which is a variation of [12, Theorem 1].

PROPOSITION 1. *Let G_1 and G_2 be graphs. Then $el(G_1 * G_2) = el(G_1 \cup G_2) = el(G_1) \cup el(G_2)$. So the elementary divisors of G are the elementary divisors of its blocks.*

Proof. Since $L(G_1 \cup G_2) = L(G_1) \dotplus (G_2)$, the second identity is clear. Number the vertices of $G_1 * G_2$ so that the first n_1 are vertices of G_1 and the last of these is the coalesced vertex. Add rows (and columns) 1 through $n_1 - 1$ and $n_1 + 1$ through $n_1 + n_2 - 1$ to row (and column) n_1 of $L(G_1 * G_2)$. The result is $A \dotplus (0) \dotplus B$, where $A \dotplus (0)$ is equivalent to $L(G_1)$ and $B \dotplus (0)$ is equivalent to $L(G_2)$. ■

A *connected sum* of G_1 and G_2 is a graph obtained from $G_1 \cup G_2$ by adding a new edge from a vertex of G_1 to a vertex of G_2. Denote by $G_1 \# G_2$ any of the $n_1 n_2$ connected sums of G_1 and G_2.

COROLLARY 1. *Let G_1 and G_2 be graphs. Then $el(G_1 \# G_2) = el(G_1 \cup$*

G_2).

Proof. This follows immediately from Proposition 1 because $G_1 \# G_2 = G_1 * P_2 * G_2$, where P_2 is the path on two vertices. ■

COROLLARY 2. *Let G be a cactus (i.e., a graph whose blocks are edges or simple circuits). Then $el(G)$ is the set of prime power factors of the lengths of the circuits in G.*

Proof. Immediate from Proposition 1 and Example 1(1.4) ■

EXAMPLE 2. Let $G = C_3 * P_7 * C_4$ and $H = C_{12}$. Both graphs have 12 vertices. From Corollary 2, $S(G) = S(H) = I_{10} \dotplus (12) \dotplus (0)$. So G and H are equivalent, yet G has 13 edges and 8 blocks, while H has 12 edges and one block.

(K. A. Berman [1, p. 7] has shown that a plane graph and its dual (multigraph) have the same invariant factors. In particular, nonisomorphic duals of the same planar graph are Laplacian equivalent.)

3. UNIMODULAR EQUIVALENCE FOR THE EDGE VERSION

The rich diversity of Laplacian equivalence contrasts sharply with the edge version:

THEOREM 1. *Let G be a connected graph with n vertices and $m > 0$ edges. Then the Smith normal form of $K(G)$ is $I_{n-2} \dotplus (n) \dotplus 0_{m-n+1}$, where the identity summand is absent when $m = 1$, and the zero summand is absent when $m = n - 1$.*

EXAMPLE 3. Let $G = C_n$. Orient the edges so that G becomes a directed cycle. Then $L(C_n) = K(C_n)$. Thus, Example 1(1.4) becomes a corollary of Theorem 1.

Proof of Theorem 1. Matrices arising from different orientations are diagonally equivalent. Hence, the Smith normal form of $K(G)$ is independent of the orientation chosen for G.

We first consider the case that $m = n - 1$, so $G = T$ is a tree. If $m = 1$, $K(T) = (2)$. So we assume $m > 1$. Since $L(T)$ and $K(T)$ share the same nonzero eigenvalues, the characteristic polynomial of $L(T)$ is just x times the characteristc polynomial of $K(T)$. It follows, using the matrix-tree theorem, that $\det K(T) = n$. The adjugate (or "classical adjoint") of $K(T)$ was determined in [8]. As a special case of that result we have the following: Let u and v be pendant vertices of T incident with edges e_i and e_j, respectively. Then $\det K_{ij} = (-1)^{d(u,v)-1}$, where K_{ij} is the $(n-2)$-by-$(n-2)$ submatrix of $K(T)$ obtained by deleting row i and column j, and $d(u, v)$ is the distance from u to v. Hence, the $(n-2)$nd determinantal divisor of $K(T)$ is 1, and the proof is complete for the case that G is a tree.

Suppose $m \geqslant n$. Number the edges of G so that the first $n-1$ of them comprise the edges of a spanning tree T of G. Then $K(T)$ is the leading $(n-1)$-by-$(n-1)$ principal submatrix of $K(G)$. So the $(n-2)$nd determinantal divisor of $K(G)$ divides 1, the $(n-2)$nd determinantal divisor of $K(T)$. Because the rank of $K(G)$ is $n-1$, it remains to show that its $(n-1)$st determinantal divisor is n. This is a consequence of Theorem 2 (below). ■

4. AN EDGE VERSION OF THE MATRIX-TREE THEOREM

Let $G = (V, E)$ be a *connected* graph with $n = o(V)$ and $m = o(E)$. Then $K(G)$ and $L(G)$ have the same rank, namely $n-1$. The matrix-tree theorem concerns determinants of $(n-1)$-square submatrices of $L(G)$. Our next result is a corresponding theorem for $K(G)$.

THEOREM 2. *Let $G = (V, E)$ be a connected graph. Let S_1 and S_2 be $(n-1)$-element subsets of E. Denote by $K[S_1 \mid S_2]$ the $(n-1)$-by-$(n-1)$ submatrix of $K(G)$ consisting of those rows corresponding to edges in S_1 and those columns corresponding to edges in S_2. Then*

$$\det K[S_1 \mid S_2] = \begin{cases} \pm n & \text{if } (V, S_1) \text{ and } (V, S_2) \text{ are both spanning trees of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $v \in V$. Let $Q(v \mid S_i]$ be the $(n-1)$-square submatrix of $Q(G)$ obtained by deleting the row corresponding to vertex v and using (keeping) the columns corresponding to the edges of S_i, $i = 1, 2$. By the Binet-Cauchy theorem.

$$\det K[S_1 \mid S_2] = \sum_{j=1}^{n} \det Q(v_j \mid S_1] \det Q(v_j \mid S_2]. \tag{1}$$

LEMMA 1. *Let $G = (V, E)$ be a connected graph on n vertices. Let $v \in V$, and let S be an $(n-1)$-element subset of E. Then $\det Q(v \mid S]$ is ± 1 if $T = (V, S)$ is a spanning tree of G, and 0 otherwise.*

This is a well-known fact. See, e.g., [2, pp. 30–33], [6, pp. 152–153], or [7, p. 144].

LEMMA 2. *Let* $G = (V, E)$ *be a connected graph with vertex set* $V = \{v_1, v_2, \dots, v_n\}$ *and edge set* E. *Let* S *be a subset of* E *such that* $T = (V, S)$ *is a spanning tree of* G. *Then either* $\det Q(v_j \mid S] = (-1)^j$ *for all* j *or* $\det Q(v_j \mid S] = (-1)^{j+1}$ *for all* j.

Proof. Consider the n-by-$(n-1)$ submatrix A of Q whose columns correspond to the edges in S. Then $AA^t = L(T)$ and $A^tA = K(T)$. Since $\det K(T) = n$, the columns of A are linearly independent. Let P be the n-by-1 matrix each of whose entries is 1. Define B as the n-by-n partitioned matrix with A occupying its first $n-1$ columns, and P its last column. Note that the nth column of B is orthogonal to the first $n-1$, so that B is invertible. Moreover, $BB^t = AA^t + J$, where J is the n-by-n matrix each of whose entries is 1. Since $L(T)P = 0$ and $JP = nP$, we conclude that the eigenvalues of BB^t are the nonzero eigenvalues of $AA^t = L(T)$ together with n. Now, the product of the nonzero eigenvalues of $L(T)$ is the coefficient of x in its characteristic polynomial, which, by the matrix-tree theorem, is n. Thus, $\det BB^t = n^2$, and $\det B = \pm n$. Expanding $\det B$ down its last column, we have

$$\sum_{j=1}^{n} (-1)^j \det Q(v_j \mid S] = \pm n.$$

Since each summand is ± 1 (Lemma 1), there is no cancellation: $\det Q(v_j \mid S] = (-1)^j$ for all j or $(-1)^{j+1}$ for all j. ∎

To complete the proof of Theorem 2, we return to (1). By Lemma 1, the sum is 0 unless both (V, S_1) and (V, S_2) are spanning trees of G. In case they are, we use Lemma 2 to conclude that for each j in (1), the product (summand) is uniformly equal to $+1$ or uniformly equal to -1. Thus, $\det K[S_1 \mid S_2] = \pm n$. ∎

5. APPLICATIONS

Let $Q = Q(G)$ be the n-by-m vertex-edge incidence matrix corresponding to some orientation of G. The *cycle space* of the oriented graph G is just the null space of Q, i.e., $C_G = \{y : Qy^t = 0\}$. The *cocycle space*, or *bond space*, of G is the row space R_G of Q. (See, e.g., [3, Chapter 12], [6, pp. 37–40], or [7, p. 144].) As subspaces of complex m-space, $C_G \cap R_G = \{0\}$. But suppose $(A, +, 0)$ is an abelian group. Then $C_G(A) = \{y \in A^m : Qy^t = 0\}$ and $R_G(A) = \{xQ : x \in A^n\}$ are subgroups of A^m, the direct product of A with itself m times.

Berman has investigated the group $B_G(A) = C_G(A) \cap R_G(A)$, calling its elements *bicycles* of G [1]. Of course, $B_G(A) = \{0\}$ if A is the additive group of integers. In other cases, the situation can be more interesting. Observe first the $B_G(A) = \{xQ : L(G)x^t = 0\}$. [Berman calls elements of $\{x \in A^n : xQ \in B_G(A)\}$ *balanced vertex weighings* of G.] If $S = S(G)$ is the Smith normal form of $L(G)$, there exist n-by-n unimodular matrices E and F such that $L(G) = ESF$. Thus, $L(G)x^t = 0$ if and only if $SFx^t = 0$. Letting x_k be the kth component of Fx^t, we obtain the equivalent conditions $s_k(G)x_k = 0$, for all k. It follows that $B_G(A)$ is isomorphic to $A(s_1(G)) \times A(s_2(G)) \times \cdots \times A(s_{n-1}(G))$, where $A(t) = \{a \in A : ta = 0\}$ [1, Theorem 2.4].

An analogous discussion for the edge version might go something like this. Let $G = (V, E)$ be an oriented (i.e., directed) graph. For each $v \in V$, let

$$v^+ = \{e \in E : v \text{ is the positive end of } e\}$$

and

$$v^- = \{e \in E : v \text{ is the negative end of } e\}.$$

For any function $f : E \to A$, define $O_f(v)$, the *output of f at v*, to be the sum over v^- of $f(e)$. Similarly, let $I_f(v)$, the *input of f at v*, be the sum over v^+ of $f(e)$. The *resultant flow* into v is $I_f(v) - O_f(v)$. For $a \in A$, we call f an *a-flow* on G if the resultant flow into v is equal to a for all $v \in V$. A *flow* is an a-flow for some $a \in A$. If A is the additive group of integers, then, of course, the only flows are 0-flows. More generally, the set $H_G(A)$ of all flows on G is a subgroup of A^m under the operation $(f + g)(e) = f(e) + g(e)$ for all $e \in E$.

Observe that $f : E \to A$ is an a-flow if and only if

$$y_i = \sum_{j=1}^{m} q_{ij} f(e_j) = a, \qquad 1 \leqslant i \leqslant n,$$

where $Q(G) = Q = (q_{ij})$ is the vertex-edge incidence matrix. But this is exactly the condition that $yQ = 0$, i.e.,

$$H_G(A) = \{f \in A^m : K(G)f^t = 0\}.$$

THEOREM 3. *Let G be a connected graph with n vertices and $m > 1$ edges. Let A be an abelian group. Then*

$$H_G(A) \cong A^{m-n+1} \times A(n).$$

Proof. By Theorem 1, there exist m-by-m unimodular matrices E and F such that $K(G) = ESF$, where $S = I_{n-2} \dotplus (n) \dotplus O_{m-n+1}$. Thus, $K(G)f^t = 0$ if and only if $SFf^t = 0$, if and only if $Ff^t \in \{0\}^{n-2} \times A(n) \times A^{m-n+1}$.

6. UNIMODULAR CONGRUENCE

Much more restricted than unimodular equivalence is unimodular congruence. Two integer matrices A and B are *congruent* if there is a *unimodular* matrix E such that $B = EAE^t$. The first person to study (Laplacian) unimodular congruence of graphs was William Watkins [12]. We rely on his work for the last result:

THEOREM 4. *Let G and H be graphs. If $L(G)$ and $L(H)$ are congruent, then G and H have the same chromatic polynomial.*

Proof. Denote the chromatic polynomial of G by

$$P_G(x) = \sum_{t=0}^{n-1} (-1)^t c_t(G) x^{n-t}.$$

Then $P_G(k)$ is the number of ways to color the vertices of G, using k colors, in which adjacent vertices are colored differently. Number the edges of G, say 1, 2, . . . , m, in a fixed but arbitrary way. For each circuit of G, delete the edge of *highest* number. The result is a broken circuit. Let $\mathscr{C}_t(G)$ be the set of those t-edged subgraphs of G which contain no broken circuits. In [14], H. Whitney proved that $c_t(G) = o(\mathscr{C}_t(G))$, the cardinality of $\mathscr{C}_t(G)$. Thus, it remains to show that $o(\mathscr{C}_t(G))$ is a congruence invariant. In fact, Watkins showed much more [12, Theorem 5]: If $L(G)$ and $L(H)$ are congruent, then G and H are cycle-isomorphic, i.e., (1) G and H have the same number, m, of edges (compare Example 2), and (2) numberings can be chosen for the edges of G and of H so that a subset of $\{1, 2, \ldots, m\}$ labels a circuit of G if and only if it labels a circuit of H. In particular, there is a *numerical bijection* between $\mathscr{C}_t(G)$ and $\mathscr{C}_t(H)$.

The converse of Theorem 4 is false. R. C. Read [10] gave a pair of "chromatically equivalent" graphs (shown in Figure 1). Since they have 128 and 120 spanning trees, respectively, they are not even Laplacian equivalent, much less Laplacian congruent.

The referee points out that Theorem 4 "follows directly from well-known results in matroid theory together with Watkins' result . . . " [13].

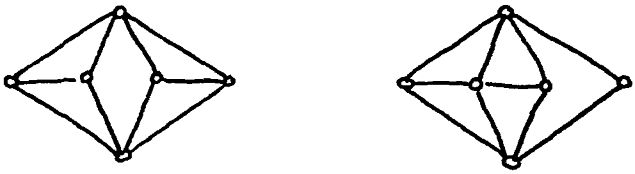

FIG. 1.

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